

Wavepacket dynamics in a quantum double-well system with the Razavy potential coupled to a harmonic oscillator

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Abstract

We have studied wavepacket dynamics in the Razavy hyperbolic double-well (DW) potential which is coupled to a harmonic oscillator (HO) by linear and quadratic interactions. Taking into account the lowest two states of DW and $(N+1)$ states of HO ($N = 1$ to 10), we evaluate eigenvalues and eigenfunctions of the composite system. An analytical calculation is made for $N = 1$ and numerical calculations are performed for $1 < N \leq 10$. Quantum tunneling of wavepackets is realized between two bottoms of composite potential $U(x, y)$ where x and y denote coordinates in DW and HO potentials, respectively. It has been shown that with increasing N and/or the coupling strength, the tunneling period is considerably increased. Phase space plots of $\langle x \rangle$ vs. $\langle p_x \rangle$ and $\langle y \rangle$ vs. $\langle p_y \rangle$ are elliptic, where $\langle \cdot \rangle$ denotes an expectation value for the two-term wavepacket. This result is quite different from the relevant one previously obtained for the quartic DW potential with the use of the quantum phase space representation [Babyuk, arXiv:0208070]. Similarity and difference between results calculated for linear and quadratic couplings, and the uncertainty relation in the model are discussed.

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I. INTRODUCTION

A study on quantum double-well (DW) systems coupled to harmonic oscillators (HOs) has been made in many fields of physics and chemistry [1]. Coupled DW plus HO systems have been investigated by using various methods such as the perturbation theory [2], time-dependent self-consistent field approximations [3], the path-integral method [4] and the quantum phase space representation [5]. Theoretical studies on this subject have conventionally adopted quartic potentials for DW systems. However, one cannot obtain exact eigenvalues and eigenfunctions of the Schrödinger equation even for quartic DW potential only (without HO). One has to apply various approximate approaches to quartic DW potential models. It is furthermore difficult to obtain definite result for the coupled DW plus HO system in which couplings between DW and HO yield an additional difficulty.

The quasi-exactly solvable hyperbolic DW potential was proposed by Razavy [6] who exactly determined a part of whole eigenvalues and eigenfunctions. A family of quasi-exactly solvable potentials has been investigated [7, 8]. In the present study, we adopt a DW system with the Razavy hyperbolic potential, which is coupled to HO. One of advantages of our adopted model is that we may use quasi-exactly solved eigenvalues and eigenfunctions of the DW system with which dynamical properties of the coupled DW plus HO system may be studied. We will consider ground and first-excited states of the DW system which are coupled with $(N + 1)$ states of HO ($N = 1 - 10$) by linear and quadratic interactions. In the case of $N = 1$, we may make exact analytical calculations of eigenvalue and eigenfunctions of the composite system, although we have to rely on numerical evaluation in the case of $N > 1$. Quite recently we have studied coupled DW systems (two qubits), each of which is described by the Razavy potential [9]. By exact calculations of eigenvalues and eigenfunctions, dynamical properties of coupled two DW systems have been successfully investigated [9]. It is worthwhile and indispensable to study wavepacket dynamics in quantum coupled DW plus HO system because it is a fundamental but unsettled subject.

The paper is organized as follows. In Sec. II, we mention the calculation method employed in our study, briefly explaining the Razavy potential [6]. Model calculations of wavepacket dynamics for linear and quadratic couplings with $N = 1$ are presented in Secs. III A and III B, respectively. In Sec. IV, we study motion of wavepackets including four terms, investigate effects of adopted model parameters on the tunneling period, and present some numerical

results for the case of $1 < N \leq 10$. The uncertainty relation in the coupled system is also studied. Sec. V is devoted to our conclusion.

II. THE ADOPTED METHOD

A. Coupled double-well system with the Razavy potential

We consider a coupled DW system whose Hamiltonian is given by

$$H = \frac{p_x^2}{2M} + V(x) + \frac{p_y^2}{2m} + \frac{m\omega^2 y^2}{2} - c x^d y, \quad (1)$$

with

$$V(x) = \frac{\hbar^2}{2M} \left[\frac{\xi^2}{8} \cosh 4x - 4\xi \cosh 2x - \frac{\xi^2}{8} \right], \quad (2)$$

where x (y) stands for coordinate of a particle of mass M (m) in DW (HO) potential; p_x (p_y) means relevant momentum; $V(x)$ signifies the Razavy DW potential [6]; ω expresses the oscillator frequency of HO; and DW and HO are coupled by linear ($d = 1$) and quadratic ($d = 2$) couplings with an interaction strength of c . The Razavy potential $V(x)$ with adopted parameters of $M = \xi = \hbar = 1.0$ is plotted in Fig. 1(a). Minima of $V(x)$ locate at $x_s = \pm 1.38433$ with $V(x_s) = -8.125$ and its maximum is $V(0) = -2.0$ at $x = 0$.

First we consider the case of $c = 0.0$ in Eq. (1). Eigenvalues of a DW system with the Razavy DW potential [Eq. (2)] are given by [6]

$$\epsilon_0 = \frac{\hbar^2}{2M} \left[-\xi - 5 - 2\sqrt{4 - 2\xi + \xi^2} \right], \quad (3)$$

$$\epsilon_1 = \frac{\hbar^2}{2M} \left[\xi - 5 - 2\sqrt{4 + 2\xi + \xi^2} \right], \quad (4)$$

$$\epsilon_2 = \frac{\hbar^2}{2M} \left[-\xi - 5 + 2\sqrt{4 - 2\xi + \xi^2} \right], \quad (5)$$

$$\epsilon_3 = \frac{\hbar^2}{2M} \left[\xi - 5 + 2\sqrt{4 + 2\xi + \xi^2} \right]. \quad (6)$$

Eigenvalues for the adopted parameters are $\epsilon_0 = -4.73205$, $\epsilon_1 = -4.64575$, $\epsilon_2 = -1.26795$ and $\epsilon_3 = 0.645751$, which lead to

$$\epsilon = \epsilon_1 + \epsilon_0 = -9.3778, \quad (7)$$

$$\delta = \epsilon_1 - \epsilon_0 = 0.0863. \quad (8)$$

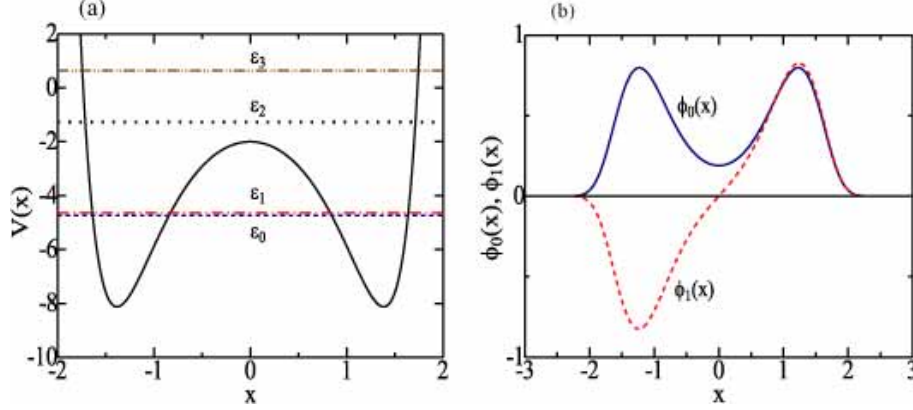


FIG. 1: (Color online) (a) The Razavy DW potential $V(x)$ [Eq.(2)] with eigenvalues of ϵ_ν ($\nu = 0 - 3$) for $\hbar = M = \xi = 1.0$. (b) Eigenfunctions of $\phi_0(x)$ (solid curve) and $\phi_1(x)$ (dashed curve).

Figure 1(a) shows that both ϵ_0 and ϵ_1 locate below $V(0)$ and that ϵ_2 and ϵ_3 are far above ϵ_1 . In this study, we take into account the lowest two states of ϵ_0 and ϵ_1 whose eigenfunctions are given by [6]

$$\phi_0(x) = A_0 e^{-\xi \cosh 2x/4} \left[3\xi \cosh x + (4 - \xi + 2\sqrt{4 - 2\xi + \xi^2}) \cosh 3x \right], \quad (9)$$

$$\phi_1(x) = A_1 e^{-\xi \cosh 2x/4} \left[3\xi \sinh x + (4 + \xi + 2\sqrt{4 + 2\xi + \xi^2}) \sinh 3x \right], \quad (10)$$

A_ν ($\nu = 0, 1$) denoting normalization factors. Figure 1(b) shows the eigenfunctions of $\phi_0(x)$ and $\phi_1(x)$, which are symmetric and anti-symmetric, respectively, with respect to the origin.

The DW system in Eq. (1) is coupled to a harmonic oscillator whose eigenfunction and eigenvalue are given by

$$\psi_n(y) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega y^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} y\right), \quad (11)$$

$$e_n = \left(n + \frac{1}{2}\right) \hbar\omega \quad (n = 0, 1, 2, \dots), \quad (12)$$

$H_n(y)$ standing for the Hermite polynomial.

B. Stationary properties

We calculate eigenvalues and eigenstates of the coupled DW system described by Eq. (1). We expand the wavefunction with basis states of $|\nu n\rangle = \phi_\nu(x)\psi_n(y)$ ($\nu = 0, 1$ and $n = 0$ to

N) as

$$\Phi(x, y) = \sum_{\nu=0}^1 \sum_{n=0}^N c_{\nu,n} \phi_{\nu}(x) \psi_n(y), \quad (13)$$

where N denotes the maximum quantum number of HO. We obtain the secular equation

$$E c_{\nu,n} = \sum_{\mu=0}^1 \sum_{k=0}^N \langle \nu n | H | \mu k \rangle c_{\mu,k}, \quad (14)$$

where

$$\begin{aligned} \langle \nu n | H | \mu k \rangle &= \left[\epsilon_{\nu} + \left(n + \frac{1}{2} \right) \hbar \omega \right] \delta_{\nu,\mu} \delta_{n,k} \\ &- [\delta_{d,1} \zeta (\delta_{\nu,\mu+1} + \delta_{\nu,\mu-1}) + \delta_{d,2} \delta_{\nu,\mu} (\zeta_0 \delta_{\nu,0} + \zeta_1 \delta_{\nu,1})] \\ &\times \left(\sqrt{n} \delta_{n,k+1} + \sqrt{n+1} \delta_{n,k-1} \right), \end{aligned} \quad (15)$$

with

$$\zeta = c \gamma \sqrt{\frac{g}{2}}, \quad (16)$$

$$\zeta_{\lambda} = c \gamma_{\lambda} \sqrt{\frac{g}{2}}, \quad \left(\lambda = 0, 1; \quad g = \sqrt{\hbar/m\omega} \right) \quad (17)$$

$$\gamma = \int_{-\infty}^{\infty} \phi_0(x) x \phi_1(x) dx = 1.13823, \quad (18)$$

$$\gamma_0 = \int_{-\infty}^{\infty} \phi_0(x) x^2 \phi_0(x) dx = 1.36128, \quad (19)$$

$$\gamma_1 = \int_{-\infty}^{\infty} \phi_1(x) x^2 \phi_1(x) dx = 1.44467. \quad (20)$$

From a diagonalization of the secular equation (14), we may obtain the eigenvalue E_{κ} and eigenfunction $\Psi_{\kappa}(x, y)$ satisfying the stationary Schrödinger equation

$$H \Phi_{\kappa}(x, y) = E_{\kappa} \Phi_{\kappa}(x, y), \quad (21)$$

where $\kappa = 0$ to $N_m = 2(N+1)-1$. Eigenvalues and eigenfunctions for $N = 1$ are analytically obtained, and those for $N > 1$ are evaluated by MATHEMATICA.

C. Dynamical properties

In the spectral method, a solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi(x, y, t)}{\partial t} = H \Psi(x, y, t), \quad (22)$$

is expressed by

$$\Psi(x, y, t) = \sum_{\kappa=0}^{N_m} a_{\kappa} \Phi_{\kappa}(x, y) e^{-iE_{\kappa}t/\hbar}, \quad (23)$$

with

$$\sum_{\kappa=0}^{N_m} |a_{\kappa}|^2 = 1, \quad (24)$$

where E_{κ} and $\Phi_{\kappa}(x, y)$ are eigenvalue and eigenfunction, respectively, obtained in Eq. (21). Expansion coefficients a_{κ} are in principle determined by a given initial wavepacket, which requires cumbersome calculations. Instead we adopt in this study, a conventional wavepacket with coefficients given by $a_0 = a_1 = 1/\sqrt{2}$ and $a_{\kappa} = 0$ for $\kappa \geq 2$,

$$\Psi(x, y, t) = \frac{1}{\sqrt{2}} [\Phi_0(x, y) e^{-iE_0t/\hbar} + \Phi_1(x, y) e^{-iE_1t/\hbar}]. \quad (25)$$

The tunneling period T for the wavepacket given by Eq. (25) is determined by

$$T = \frac{2\pi\hbar}{E_1 - E_0} = \frac{2\pi}{\Omega_1}, \quad (26)$$

where $\Omega_1 = (E_1 - E_0)/\hbar$. We will study a wavepacket with $a_0 = a_1 = a_2 = a_3 = 1/2$ in Sec. IV A, whose tunneling period is not given by Eq. (26).

III. MODEL CALCULATIONS

Introducing a parameter α , we express the harmonic oscillator frequency ω by

$$\hbar\omega = \alpha(\epsilon_1 - \epsilon_0) = \alpha\delta. \quad (27)$$

Coefficients of ζ , ζ_0 and ζ_1 in Eqs. (16) and (17) are expressed in terms of m , α and c as follows:

$$\zeta = c\gamma \left(\frac{\hbar^2}{4m\alpha\delta} \right)^{1/4} = c \left(\frac{\gamma\sqrt{\hbar}}{\sqrt{2}\delta^{1/4}} \right) \left(\frac{1}{m\alpha} \right)^{1/4} = 1.485 c \left(\frac{1}{m\alpha} \right)^{1/4}, \quad (28)$$

$$\zeta_0 = c \left(\frac{\gamma_0\sqrt{\hbar}}{\sqrt{2}\delta^{1/4}} \right) \left(\frac{1}{m\alpha} \right)^{1/4} = 1.776 c \left(\frac{1}{m\alpha} \right)^{1/4}, \quad (29)$$

$$\zeta_1 = c \left(\frac{\gamma_1\sqrt{\hbar}}{\sqrt{2}\delta^{1/4}} \right) \left(\frac{1}{m\alpha} \right)^{1/4} = 1.885 c \left(\frac{1}{m\alpha} \right)^{1/4}. \quad (30)$$

Table 1 summarizes various coefficients appearing in our model calculations. δ , γ , γ_0 , γ_1 and η are determined by the Razavy potential with $M = \xi = 1.0$ whereas γ_y and η_y are given by HO potential with $m = 1.0$ and $\omega = \alpha\delta$ ($\alpha = 10.0$). Then model calculations to be reported will be specified by a set of parameters of m , α , c and N .

Coefficient	Definition	Value	Note
δ	$\epsilon_1 - \epsilon_0$	0.08630	Eq. (8)
γ	$\langle \phi_0 x \phi_1 \rangle_x$	1.1382	Eq. (18)
γ_0	$\langle \phi_0 x^2 \phi_0 \rangle_x$	1.3613	Eq. (19)
γ_1	$\langle \phi_1 x^2 \phi_1 \rangle_x$	1.4447	Eq. (20)
η	$\langle \phi_0 \partial_x \phi_1 \rangle_x$	0.09823	Eq. (56)
γ_y	$\langle \psi_0 y \psi_1 \rangle_y$	0.76117	Eq. (57)
η_y	$\langle \psi_0 \partial_y \psi_1 \rangle_y$	0.65689	Eq. (58)

Table 1 Various coefficients in model calculations with $M = \xi = m = \hbar = 1.0$ and $\alpha = 10.0$, $\langle \cdot \rangle_x$ and $\langle \cdot \rangle_y$ denoting integrals over x and y , respectively (see text).

Figures 2(a) and 2(a) show contour maps of the composite potential $U(x, y) = \mu$ defined by

$$U(x, y) = V(x) + \frac{m\omega^2 y^2}{2} - c x^d y, \quad (31)$$

for linear ($d = 1$) and quadratic ($d = 2$) couplings, respectively, with $c = 0.0$ (dashed curves) and $c = 1.0$ (solid curves) for $\mu = -5.0, 0.0, 5.0$ and 10.0 ($m = 1.0$ and $\alpha = 10.0$). For $c = 0$, $U(x, y)$ has two minima of $U(x, y) = -8.125$ at $(x, y) = (\pm 1.3843, 0.0)$. For a linear coupling ($d = 1$) with $c = 1.0$, it has two minima of $U(x, y) = -9.438$ at $(x, y) = (1.4120, 1.8959)$ and $(-1.4120, -1.8959)$. For a quadratic coupling ($d = 2$) with $c = 1.0$, it has two minima of $U(x, y) = -9.125$ at $(x, y) = (1.4120, 1.8959)$ and $(-1.4120, 1.8959)$. Model calculations for linear and quadratic couplings with $N = 1$ will be separately reported in Secs. III A and III B, respectively.

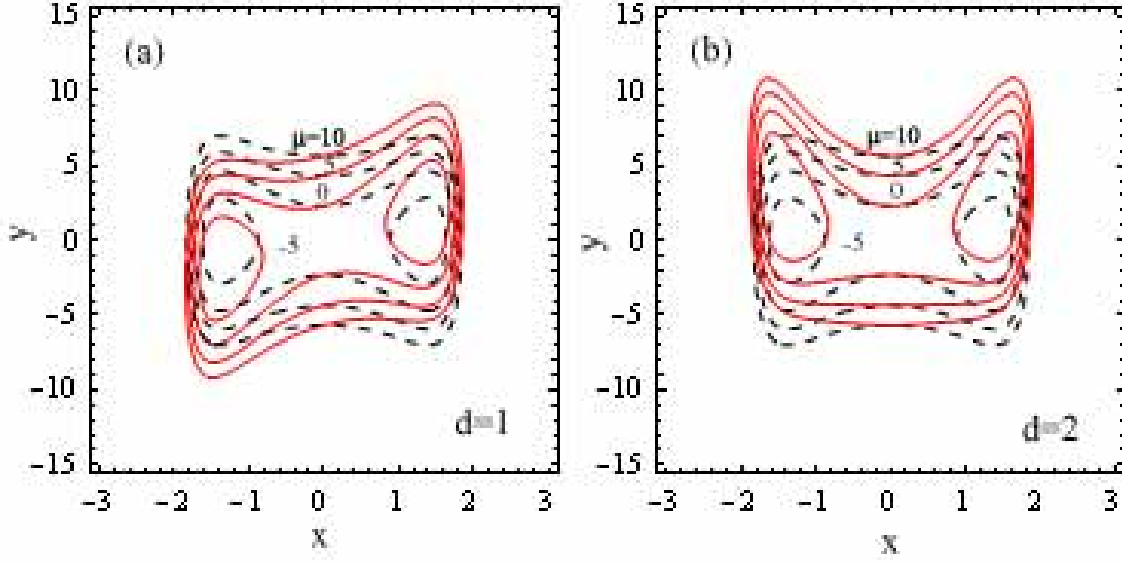


FIG. 2: (Color online) Contour plots of $U(x, y) = \mu$ with $\mu = -5, 0, 5$ and 10 (from the inside) for (a) linear ($d = 1$) and (b) quadratic ($d = 2$) couplings with $c = 0.0$ (dashed curves) and $c = 1.0$ (solid curves).

A. Linear coupling with $N = 1$

For a linear coupling ($d = 1$) with $N = 1$, the energy matrix of the Hamiltonian given by Eq. (1) is expressed in the basis of $\psi_0(y)\phi_0(x)$, $\psi_0(y)\phi_1(x)$, $\psi_1(y)\phi_0(x)$ and $\psi_1(y)\phi_1(x)$ by

$$\mathcal{H} = \begin{pmatrix} \epsilon_0 + \hbar\omega/2 & 0 & 0 & -\zeta \\ 0 & \epsilon_1 + \hbar\omega/2 & -\zeta & 0 \\ 0 & -\zeta & \epsilon_0 + 3\hbar\omega/2 & 0 \\ -\zeta & 0 & 0 & \epsilon_1 + 3\hbar\omega/2 \end{pmatrix}, \quad (32)$$

where ζ is given by Eq. (28). We obtain eigenvalues of the energy matrix

$$E_0 = \frac{\epsilon}{2} + \hbar\omega - \sqrt{\frac{1}{4}(\hbar\omega + \delta)^2 + \zeta^2}, \quad (33)$$

$$E_1 = \frac{\epsilon}{2} + \hbar\omega - \sqrt{\frac{1}{4}(\hbar\omega - \delta)^2 + \zeta^2}, \quad (34)$$

$$E_2 = \frac{\epsilon}{2} + \hbar\omega + \sqrt{\frac{1}{4}(\hbar\omega - \delta)^2 + \zeta^2}, \quad (35)$$

$$E_3 = \frac{\epsilon}{2} + \hbar\omega + \sqrt{\frac{1}{4}(\hbar\omega + \delta)^2 + \zeta^2}. \quad (36)$$

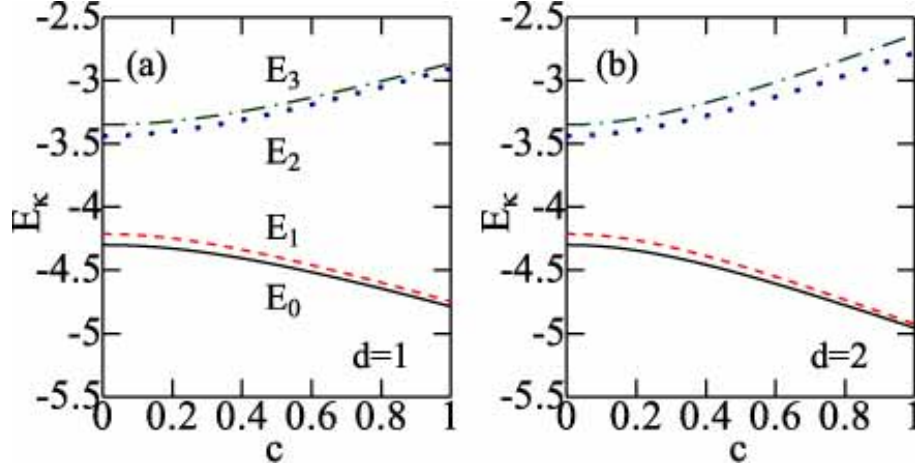


FIG. 3: (Color online) The c dependence of eigenvalues E_κ ($\kappa = 0-3$) of (a) a linear coupling ($d = 1$) and (b) a quadratic coupling ($d = 2$) with $m = 1.0$, $\alpha = 10.0$ and $N = 1$.

Relevant eigenfunctions are expressed by

$$\Phi_0(x, y) = \cos \theta_1 \psi_0(y) \phi_0(x) + \sin \theta_1 \psi_1(y) \phi_1(x), \quad (37)$$

$$\Phi_1(x, y) = \cos \theta_2 \psi_0(y) \phi_1(x) + \sin \theta_2 \psi_1(y) \phi_0(x), \quad (38)$$

$$\Phi_2(x, y) = -\sin \theta_2 \psi_0(y) \phi_1(x) + \cos \theta_2 \psi_1(y) \phi_0(x), \quad (39)$$

$$\Phi_3(x, y) = -\sin \theta_1 \psi_0(y) \phi_0(x) + \cos \theta_1 \psi_1(y) \phi_1(x), \quad (40)$$

where

$$\tan 2\theta_1 = \frac{2\zeta}{(\hbar\omega + \delta)}, \quad (41)$$

$$\tan 2\theta_2 = \frac{2\zeta}{(\hbar\omega - \delta)}. \quad (42)$$

Eigenvalues E_κ ($\kappa = 0 - 3$) for $d = 1$ with $m = 1.0$ and $\alpha = 10.0$ are plotted as a function of c in Fig. 3(a): Fig. 3(b) for $d = 2$ will be explained later (Sec. III B). An energy gap between the ground and first-excited states is $\Omega_1 = 0.08630$ for $c = 0.0$ and $\Omega_1 = 0.03958$ for $c = 1.0$. Ω_1 is decreased with increasing c . Figures 4(a)-4(d) show 3D plots of eigenfunctions of $\Phi_\kappa(x, y)$ ($\kappa = 0 - 3$).

We investigate motion of a wavepacket consisting of $\Phi_0(x, y)$ and $\Phi_1(x, y)$ given by Eq. (25). Time-dependent wavepackets are illustrated in Figs. 5(a)-(f) which show 3D plots of $|\Psi(x, y, t)|^2$ at (a) $t = 0.0$, (b) $0.1T$, (c) $0.2T$, (d) $0.3T$, (e) $0.4T$ and (f) $0.5T$, where $T = 158.73$ obtained by $\Omega_1 = 0.03958$. Wavepackets at $t = 0.6T, 0.7T, 0.8T, 0.9T$ and T

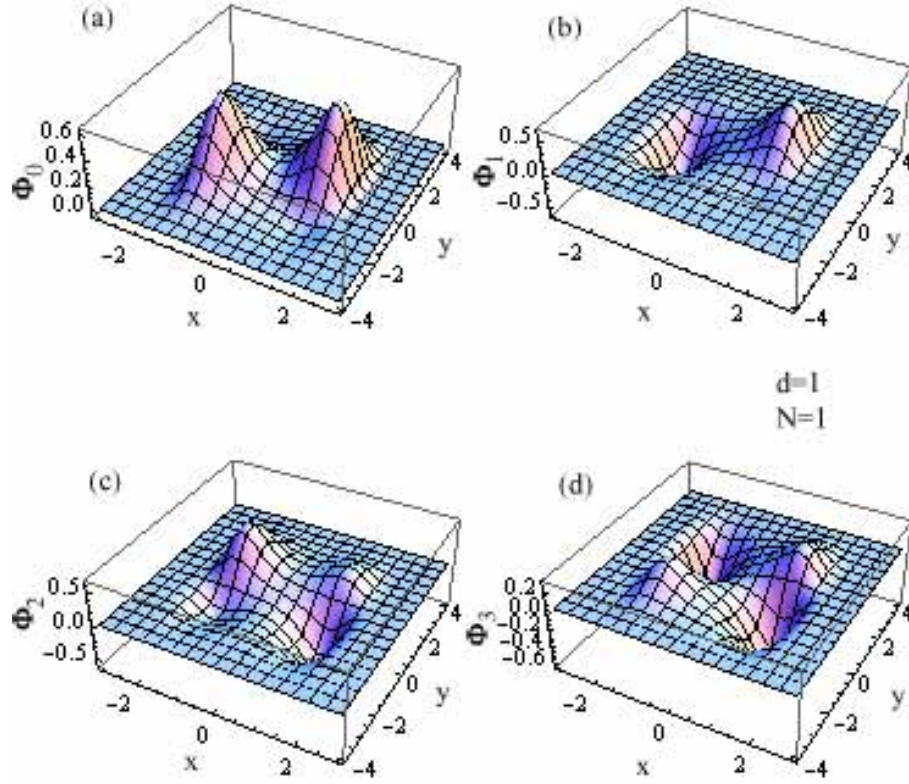


FIG. 4: (Color online) Eigenfunctions of (a) $\Phi_0(x, y)$, (b) $\Phi_1(x, y)$, (c) $\Phi_2(x, y)$, and (d) $\Phi_3(x, y)$ for a linear coupling ($d = 1$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

are the same as those at $t = 0.4T$, $0.3T$, $0.2T$, $0.1T$ and 0.0 , respectively. At $t = 0.0$, a peak of the wavepacket locates at $(x_m, y_m) = (1.2353, 0.61990)$. With time developing, a peak of the wavepacket at $(x, y) = (-1.2353, -0.61990)$ is growing, and it goes back to the initial position at $t = T$. The wavepacket shows a tunneling from $(x, y) = (1.2353, 0.61990)$ to $(x, y) = (-1.2353, -0.61990)$ across the potential barrier at the origin [see Fig. 2(a)].

By using Eqs. (25), (37)-(40), we may calculate marginal probability densities of x and

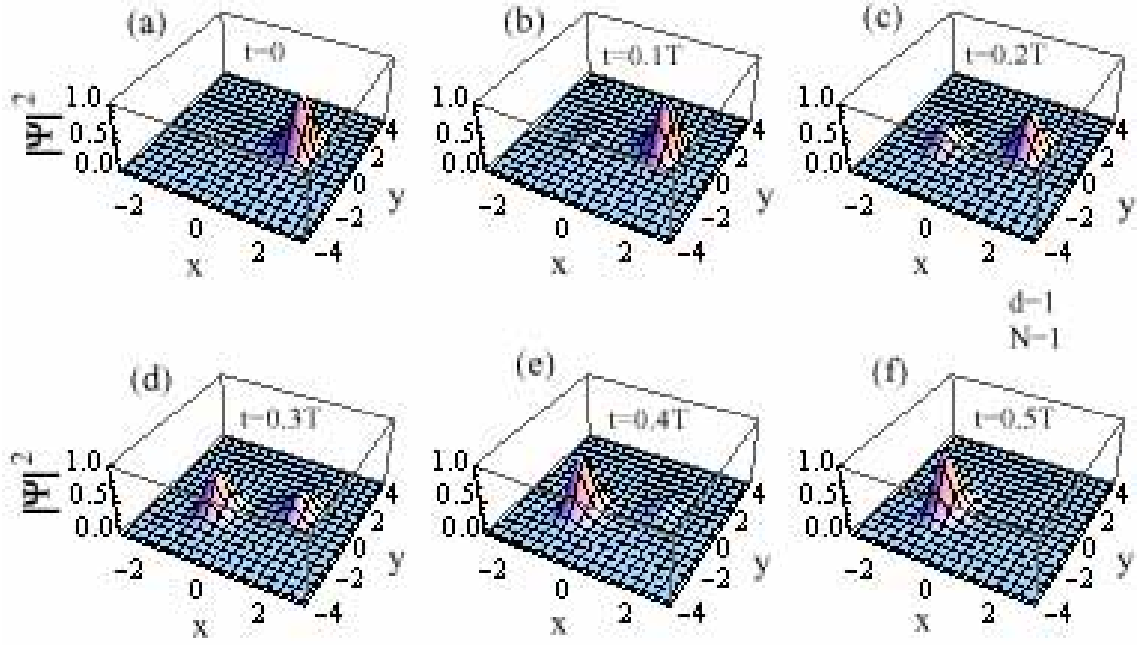


FIG. 5: (Color online) $|\Psi(x, y, t)|^2$ for a linear coupling ($d = 1$) at (a) $t = 0.0$, (b) $t = 0.1T$, (c) $t = 0.2T$, (d) $t = 0.3T$, (e) $t = 0.4T$ and (f) $t = 0.5T$ where $T = 158.73$ ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

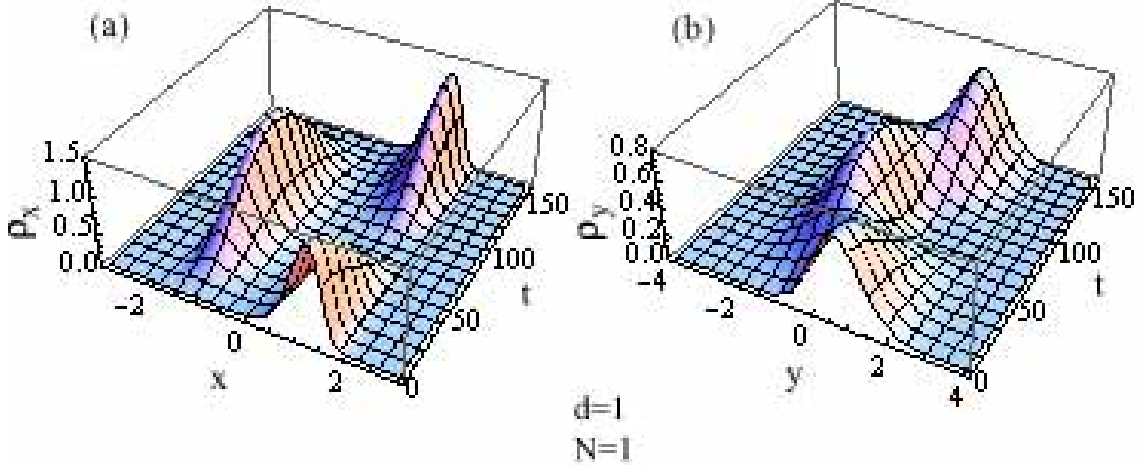


FIG. 6: (Color online) Time dependence of (a) $\rho_x(t)$ and (b) $\rho_y(t)$ for a linear coupling ($d = 1$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

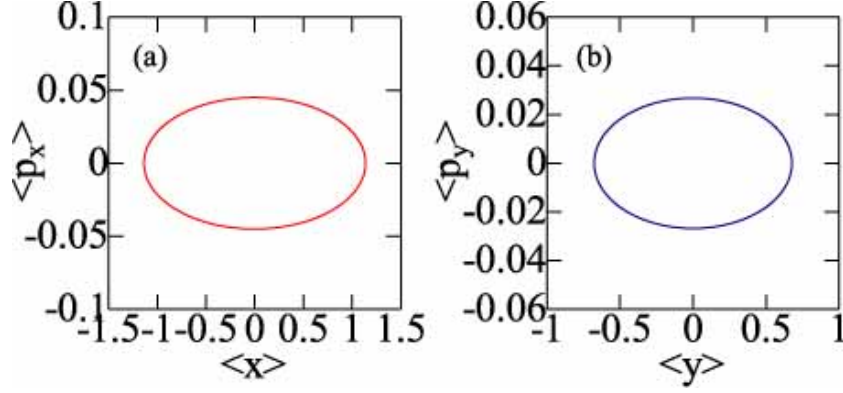


FIG. 7: (Color online) Phase space representations of (a) $\langle p_x \rangle$ vs. $\langle x \rangle$ and (b) $\langle p_y \rangle$ vs. $\langle y \rangle$ for a linear coupling ($d = 1$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

y components, which are given by

$$\rho_x(t) = \int_{-\infty}^{\infty} |\Psi(x, y, t)|^2 dy, \quad (43)$$

$$= \frac{1}{2} (\cos^2 \theta_1 + \sin^2 \theta_2) \phi_0(x)^2 + \frac{1}{2} (\sin^2 \theta_1 + \cos^2 \theta_2) \phi_1(x)^2 + \cos(\theta_1 - \theta_2) \phi_0(x) \phi_1(x) \cos \Omega_1 t, \quad (44)$$

$$\rho_y(t) = \int_{-\infty}^{\infty} |\Psi(x, y, t)|^2 dx, \quad (45)$$

$$= \frac{1}{2} (\cos^2 \theta_1 + \cos^2 \theta_2) \psi_0(y)^2 + \frac{1}{2} (\sin^2 \theta_1 + \sin^2 \theta_2) \psi_1(y)^2 + \sin(\theta_1 + \theta_2) \psi_0(y) \psi_1(y) \cos \Omega_1 t. \quad (46)$$

Figures 6(a) and 6(b) show $\rho_x(t)$ and $\rho_y(t)$, respectively. Both $\rho_x(t)$ and $\rho_y(t)$ oscillate with the same period.

The tunneling probability of $P_r(t)$ for finding a particle in the negative x region is given by

$$P_r(t) = \int_{-\infty}^0 \rho_x(t) dx, \quad (47)$$

$$= \frac{1}{2} - b \cos(\theta_1 - \theta_2) \cos \Omega_1 t, \quad (48)$$

with

$$b = - \int_{-\infty}^0 \phi_0(x) \phi_1(x) dx = 0.496213. \quad (49)$$

By simple calculations, we obtain various time-dependent expectation values given by

$$\langle x \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^*(x, y, t) x \Psi(x, y, t) dx dy, \quad (50)$$

$$= \gamma \cos(\theta_1 - \theta_2) \cos \Omega_1 t, \quad (51)$$

$$\langle p_x \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^*(x, y, t) (-i\partial_x) \Psi(x, y, t) dx dy, \quad (52)$$

$$= -\eta \cos(\theta_1 + \theta_2) \sin \Omega_1 t, \quad (53)$$

$$\langle y \rangle = \gamma_y \sin(\theta_1 + \theta_2) \cos \Omega_1 t, \quad (54)$$

$$\langle p_y \rangle = \eta_y \sin(\theta_1 - \theta_2) \sin \Omega_1 t, \quad (55)$$

with

$$\eta = \int_{-\infty}^{\infty} \phi_0(x) \partial_x \phi_1(x) dx = - \int_{-\infty}^{\infty} \phi_1(x) \partial_x \phi_0(x) dx = 0.09823, \quad (56)$$

$$\gamma_y = \int_{-\infty}^{\infty} \psi_0(y) y \psi_1(y) dy = 0.76117, \quad (57)$$

$$\eta_y = \int_{-\infty}^{\infty} \psi_0(y) \partial_y \psi_1(y) dy = - \int_{-\infty}^{\infty} \psi_1(y) \partial_y \psi_0(y) dy = 0.65689, \quad (58)$$

where γ is given by Eq. (18). We generally observe that $\langle p_x \rangle = M \langle dx/dt \rangle \neq M d\langle x \rangle/dt$ because of the nonlinearity of the adopted system. Parametric plots of both $\langle x \rangle$ vs. $\langle p_x \rangle$ and $\langle y \rangle$ vs. $\langle p_y \rangle$ are elliptic, as shown in Figs. 7(a) and 7(b).

B. Quadratic coupling with $N = 1$

Next we consider a quadratic coupling ($d = 2$), for which the energy matrix with $N = 1$ is expressed in the basis of $\psi_0(y)\phi_0(x)$, $\psi_0(y)\phi_1(x)$, $\psi_1(y)\phi_0(x)$ and $\psi_1(y)\phi_1(x)$ by

$$\mathcal{H} = \begin{pmatrix} \epsilon_0 + \hbar\omega/2 & 0 & -\zeta_0 & 0 \\ 0 & \epsilon_1 + \hbar\omega/2 & 0 & -\zeta_1 \\ -\zeta_0 & 0 & \epsilon_0 + 3\hbar\omega/2 & 0 \\ 0 & -\zeta_1 & 0 & \epsilon_1 + 3\hbar\omega/2 \end{pmatrix}, \quad (59)$$

where ζ_0 and ζ_1 are given by Eqs. (29) and (30), respectively. We obtain eigenvalues of the energy matrix given by

$$E_0 = \epsilon_0 + \hbar\omega - \sqrt{\frac{1}{4}(\hbar\omega)^2 + \zeta_0^2}, \quad (60)$$

$$E_1 = \epsilon_1 + \hbar\omega - \sqrt{\frac{1}{4}(\hbar\omega)^2 + \zeta_1^2}, \quad (61)$$

$$E_2 = \epsilon_0 + \hbar\omega + \sqrt{\frac{1}{4}(\hbar\omega)^2 + \zeta_0^2}, \quad (62)$$

$$E_3 = \epsilon_1 + \hbar\omega + \sqrt{\frac{1}{4}(\hbar\omega)^2 + \zeta_1^2}. \quad (63)$$

Relevant eigenfunctions are expressed by

$$\Phi_0(x, y) = \cos \theta_1 \psi_0(y) \phi_0(x) + \sin \theta_1 \psi_1(y) \phi_0(x), \quad (64)$$

$$\Phi_1(x, y) = \cos \theta_2 \psi_0(y) \phi_1(x) + \sin \theta_2 \psi_1(y) \phi_1(x), \quad (65)$$

$$\Phi_2(x, y) = -\sin \theta_1 \psi_0(y) \phi_0(x) + \cos \theta_1 \psi_1(y) \phi_0(x), \quad (66)$$

$$\Phi_3(x, y) = -\sin \theta_2 \psi_0(y) \phi_1(x) + \cos \theta_2 \psi_1(y) \phi_1(x), \quad (67)$$

where

$$\tan 2\theta_1 = \frac{2\zeta_0}{\hbar\omega}, \quad (68)$$

$$\tan 2\theta_2 = \frac{2\zeta_1}{\hbar\omega}. \quad (69)$$

Eigenvalues E_κ ($\kappa = 0-3$) for $d = 2$ with $m = 1.0$ and $\alpha = 10.0$ are plotted as a function of c in Fig. 3(b). An energy gap between the ground and first-excited states is $\Omega_1 = 0.08630$ and 0.02989 for $c = 0.0$ and 1.0 , respectively. The c dependence of eigenvalues for a quadratic coupling is similar to that for a linear coupling shown in Fig. 3(a). Figures 8(a)-8(d) show eigenfunctions of $\Phi_\kappa(x, y)$ ($\kappa = 0-3$).

We consider a wavepacket $\Psi(x, y, t)$ given by Eq. (25). Figures 9(a)-9(f) show 3D plots of $|\Psi(x, y, t)|^2$ at (a) $T = 0$, (b) $0.1T$, (c) $0.2T$, (d) $0.3T$, (e) $0.4T$ and (f) $0.5T$ where $T = 210.25$. At $t = 0.0$, a peak of the wavepacket locates at $(x_m, y_m) = (1.2354, 0.6468)$. At $t \sim 0.5 T$, wavepacket has appreciable magnitude at $(x, y) = (-1.2354, 0.6468)$ because the minimum of the composite potential $U(x, y)$ locates at $(x, y) = (-1.4120, 1.8959)$. The tunneling of the wavepacket occurs between $(x, y) = (1.2354, 0.6468)$ and $(-1.2354, 0.6468)$.

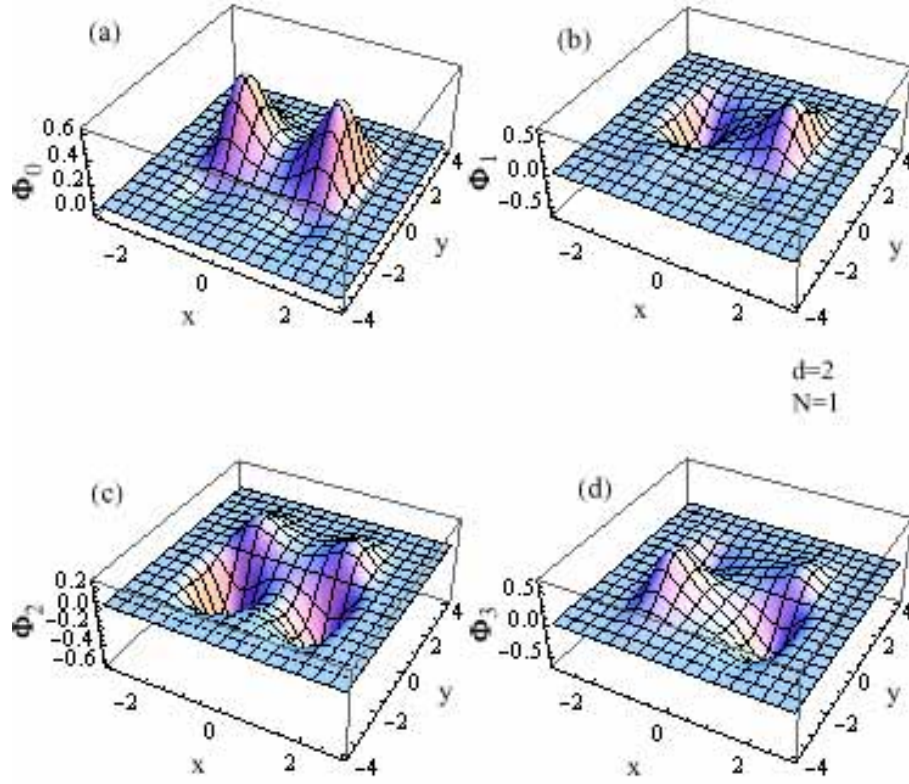


FIG. 8: (Color online) Eigenfunctions of (a) $\Phi_0(x, y)$, (b) $\Phi_1(x, y)$, (c) $\Phi_2(x, y)$, and (d) $\Phi_3(x, y)$ for the quadratic coupling ($d = 2$) for a quadratic coupling ($d = 2$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

By using Eqs. (25), (37)-(40), we may obtain marginal probability densities of x and y components, which are given by

$$\rho_x(t) = \frac{1}{2} [\phi_0(x)^2 + \phi_1(x)^2] + \cos(\theta_1 - \theta_2) \phi_0(x) \phi_1(x) \cos \Omega_1 t, \quad (70)$$

$$\begin{aligned} \rho_y(t) = & \frac{1}{2} [(\cos^2 \theta_1 + \cos^2 \theta_2) \psi_0(y)^2 + (\sin^2 \theta_1 + \sin^2 \theta_2) \psi_1(y)^2] \\ & + \frac{1}{2} [\sin(2\theta_1) + \sin(2\theta_2) \psi_0(y) \psi_1(y)]. \end{aligned} \quad (71)$$

Figures 10(a) and 10(b) show $\rho_x(t)$ and $\rho_y(t)$, respectively. $\rho_x(t)$ is similar to the relevant result for the linear coupling in Fig. 6(a) although $\rho_y(t)$ is different from that in Fig. 6(b).

The tunneling probability $P_r(t)$ is given by

$$P_r(t) = \frac{1}{2} - b \cos(\theta_1 - \theta_2) \cos \Omega_1 t, \quad (72)$$

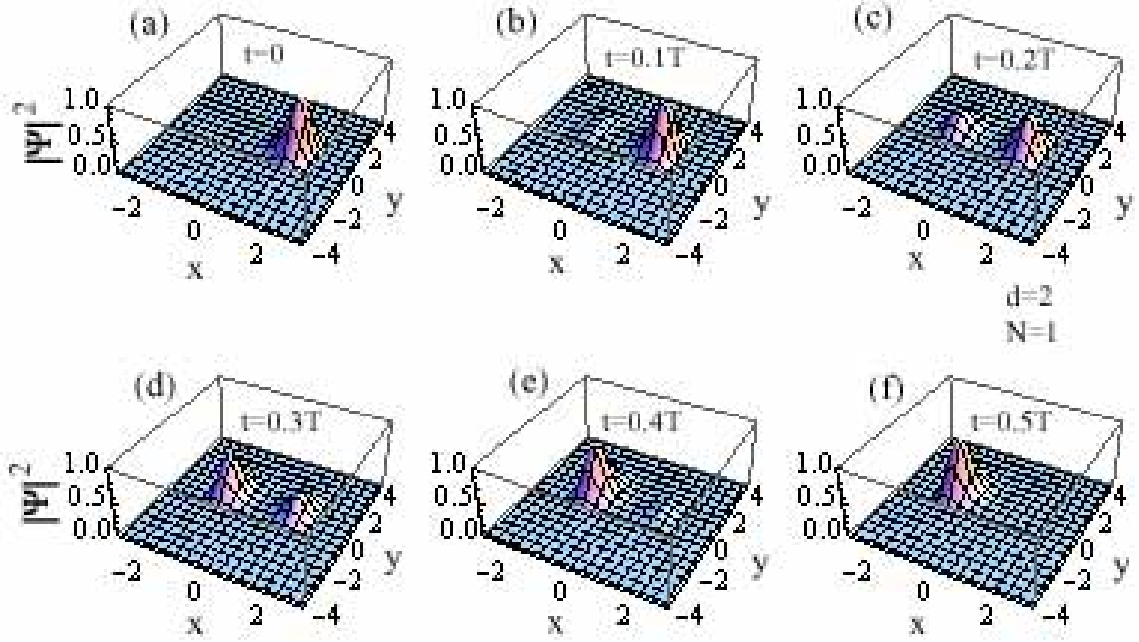


FIG. 9: (Color online) $|\Psi(x, y, t)|^2$ for a quadratic coupling ($d = 2$) at (a) $t = 0.0$, (b) $t = 0.1T$, (c) $t = 0.2T$, (d) $t = 0.3T$, (e) $t = 0.4T$ and (f) $t = 0.5T$ where $T = 210.25$ for a quadratic coupling ($d = 2$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

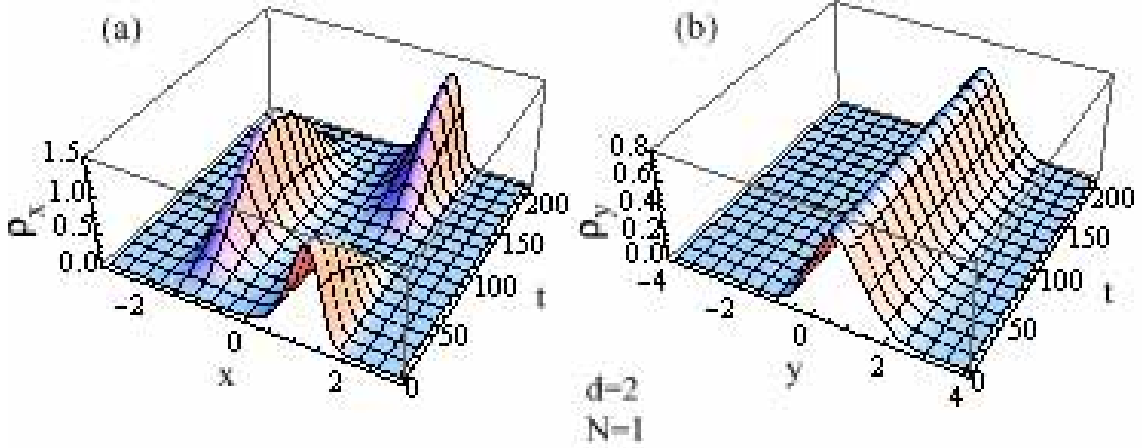


FIG. 10: (Color online) Time dependence of (a) $\rho_x(t)$ and (b) $\rho_y(t)$ for a quadratic coupling ($d = 2$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

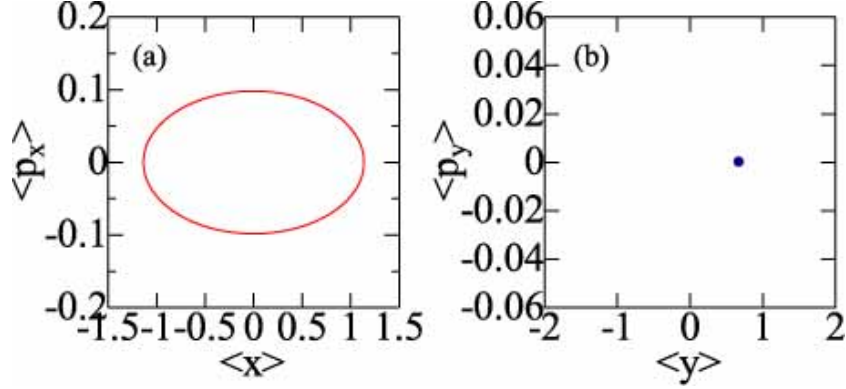


FIG. 11: (Color online) Phase space representations of (a) $\langle p_x \rangle$ vs. $\langle x \rangle$ and (b) $\langle p_y \rangle$ vs. $\langle y \rangle$ for a quadratic coupling ($d = 2$) ($m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$).

which is the same as Eq. (48) for a linear coupling.

Various time-dependent expectation values are given by

$$\langle x \rangle = \gamma \cos(\theta_1 - \theta_2) \cos \Omega_1 t, \quad (73)$$

$$\langle p_x \rangle = -\eta \cos(\theta_1 - \theta_2) \sin \Omega_1 t, \quad (74)$$

$$\langle y \rangle = \frac{\gamma_y}{2} (\sin 2\theta_1 + \sin 2\theta_2), \quad (75)$$

$$\langle p_y \rangle = 0. \quad (76)$$

Figures 11(a) and 11(b) show parametric plots of $\langle x \rangle$ vs. $\langle p_x \rangle$ and $\langle y \rangle$ vs. $\langle p_y \rangle$, respectively. The former is ellipsoid while the latter is a point at $(\langle y \rangle, \langle p_y \rangle) = (0.70186, 0.0)$ staying at the initial state. Although $\langle x \rangle$ vs. $\langle p_x \rangle$ plot in Fig. 11(a) is similar to that for a linear coupling in Fig. 7(a), $\langle y \rangle$ vs. $\langle p_y \rangle$ plot in Fig. 11(b) is quite different from that for a linear coupling in Fig. 7(b).

IV. DISCUSSION

A. A wavepacket with $a_0 = a_1 = a_2 = a_3 = 1/2$

In the preceding section, we consider a wavepacket with $a_0 = a_1 = 1/\sqrt{2}$ and $a_2 = a_3 = 0.0$. Here we will study a four-component wavepacket with coefficients of $a_0 = a_1 = a_2 =$

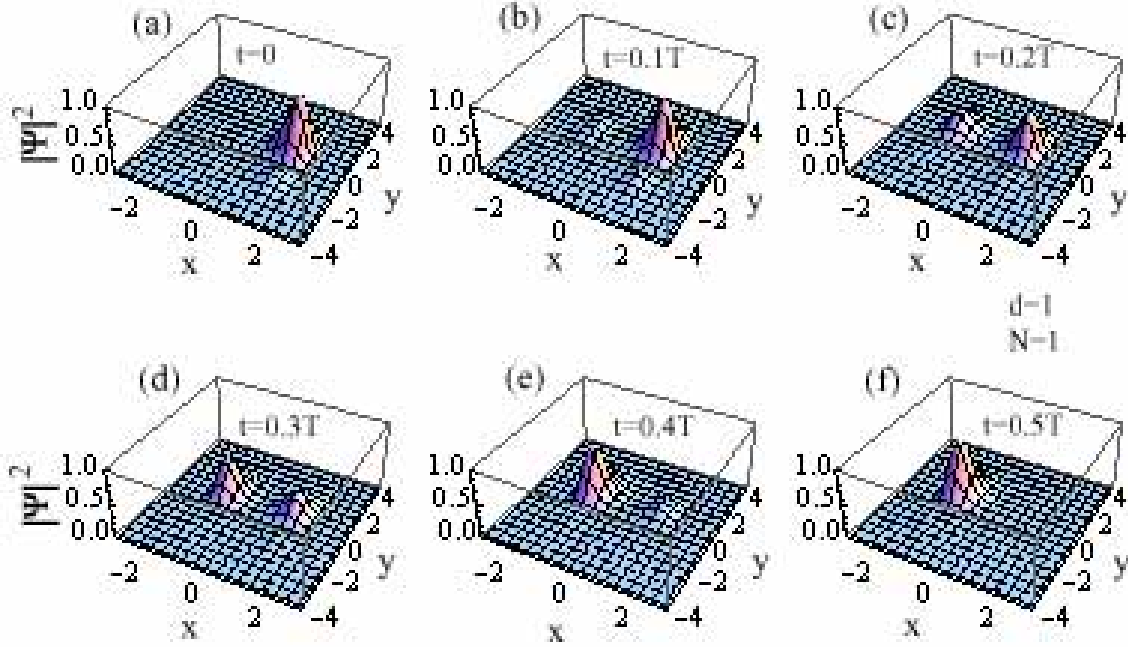


FIG. 12: (Color online) $|\Psi(x, y, t)|^2$ of the four-component wavepacket given by Eq. (77) for a linear coupling ($d = 1$) at (a) $t = 0.0$, (b) $t = 0.1T$, (c) $t = 0.2T$, (d) $t = 0.3T$, (e) $t = 0.4T$ and (f) $t = 0.5T$ where $T = 71.6084$ ($m = 1.0$, $\alpha = 10.0$, $c = 0.1$ and $N = 1$).

$a_3 = 1/2$ in Eq. (23)

$$\Psi(x, y, t) = \frac{1}{2} \sum_{\kappa=0}^3 \Phi_{\kappa}(x, y) e^{-iE_{\kappa}t/\hbar}. \quad (77)$$

For a linear coupling ($d = 1$) with $c = 0.1$, $m = 1.0$, $\alpha = 10.0$ and $N = 1$, we obtain eigenvalues of $(E_0, E_1, E_2, E_3) = (-4.30784, -4.22313, -3.42868, -3.34397)$. The peak of the wavepacket initially locates at $(x, y) = (x_m, y_m) = (1.2353, 0.80775)$. The time-dependence of $|\Psi(x, y, t)|^2$ from $t = 0$ to $t = T/2$ are shown in Fig. 12 where $T = 71.6084$ (below). With time developing, a new peak appears at $(x, y) \neq (x_m, y_m)$, and at $t = T$ wavepacket returns to its initial position.

Calculations of $\rho_x(t)$ and $\rho_y(t)$ for this wavepacket consisting of four terms are very tedious though it is not impossible. As their substitutes, we show the 3D plot of $|\Psi(x, y_m, t)|^2$ as functions of x and t in Fig. 13(a), and that of $|\Psi(x_m, y, t)|^2$ as functions of y and t in Fig. 13(b). Both $|\Psi(x, y_m, t)|^2$ and $|\Psi(x_m, y, t)|^2$ show complicated and rapid oscillations. The dashed curve in Fig. 14 expresses $|\Psi(x_m, y_m, t)|^2$ as a function of t , and the solid curve shows

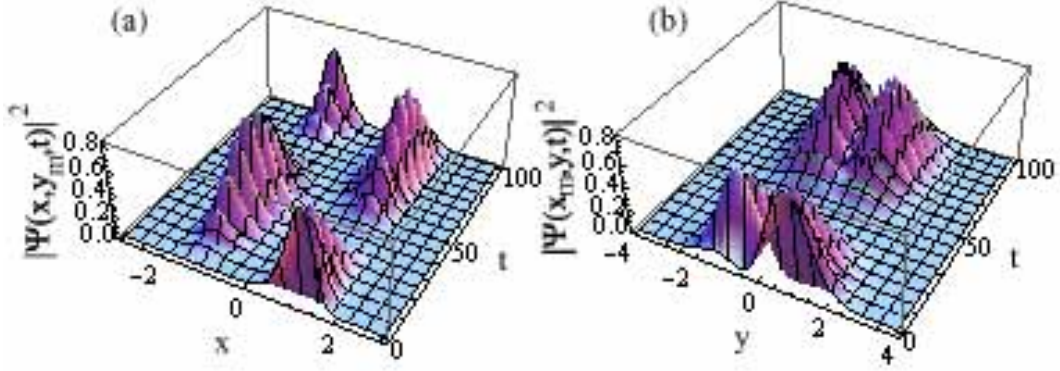


FIG. 13: (Color online) (a) 3D plots of $|\Psi(x, y_m, t)|^2$ with $y_m = 0.761169$ and (b) $|\Psi(x_m, y, t)|^2$ with $x_m = 1.23534$ of the wavepacket given by Eq. (77) ($m = 1.0$, $\alpha = 10.0$, $c = 0.1$, $d = 1$ and $N = 1$).

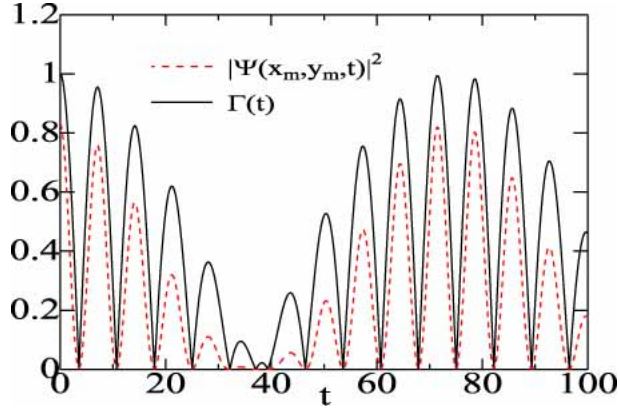


FIG. 14: (Color online) Time dependence of $\Gamma(t)$ (solid curve) and $|\Psi(x_m, y_m, t)|^2$ (dashed curve) with $(x_m, y_m) = (1.23534, 0.761169)$ for the wavepacket given by Eq. (77) ($m = 1.0$, $\alpha = 10.0$, $c = 0.1$, $d = 1$ and $N = 1$).

the correlation function $\Gamma(t)$ defined by

$$\Gamma(t) = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi^*(x, y, 0) \Psi(x, y, t) dx dy \right|, \quad (78)$$

$$= \frac{1}{4} |1 + e^{-i\Omega_1 t} + e^{-i\Omega_2 t} + e^{-i\Omega_3 t}|. \quad (\Omega_\kappa = (E_\kappa - E_0)/\hbar) \quad (79)$$

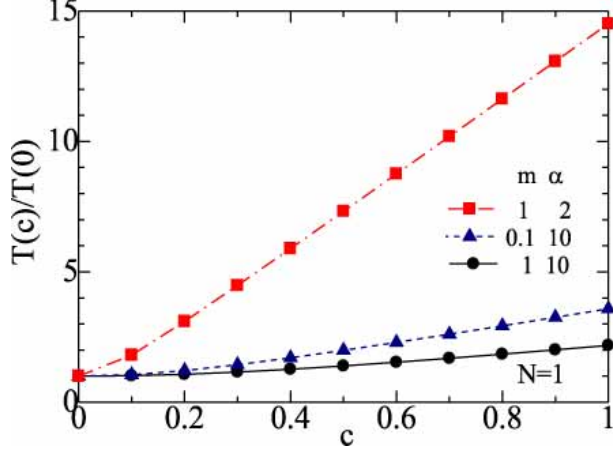


FIG. 15: (Color online) The c dependence of the tunneling period T for three sets of parameters of $(m, \alpha) = (1.0, 10.0)$ (solid curve), $(0.1, 10.0)$ (dashed curve) and $(1.0, 2.0)$ (chain curve) with $N = 1$.

From the condition for the tunneling period T ,

$$T = \min_{\forall t > 0} \{ \Gamma(t) = 1 \}, \quad (80)$$

we obtain $T = 71.6084$ which is slightly different from a value estimated by $2\pi/(E_1 - E_0) = 74.1706$. Although the tunneling period is mainly determined by E_0 and E_1 , its precise value is influenced by contributions from higher excited states with E_2 and E_3 .

B. Coupling dependence of T for other choices of parameters of m and α

We have so far presented model calculations with a set of parameters of $(m, \alpha) = (1.0, 10.0)$ with $c = 1.0$ and $N = 1$. We have calculated the tunneling period T as a function of the interaction c for three sets of parameters: $(m, \alpha) = (1.0, 10.0)$, $(0.1, 10.0)$ and $(1.0, 2.0)$, whose results are shown in Fig. 15. We note that T is increased with increasing c , which is more significant for smaller m and for smaller α .

C. The case of $N > 1$

In the case of $N > 1$, we have to numerically evaluate eigenvalues and eigenfunctions of the energy matrix with dimension of $(N_m + 1) \times (N_m + 1)$ by using MATHEMATICA, where

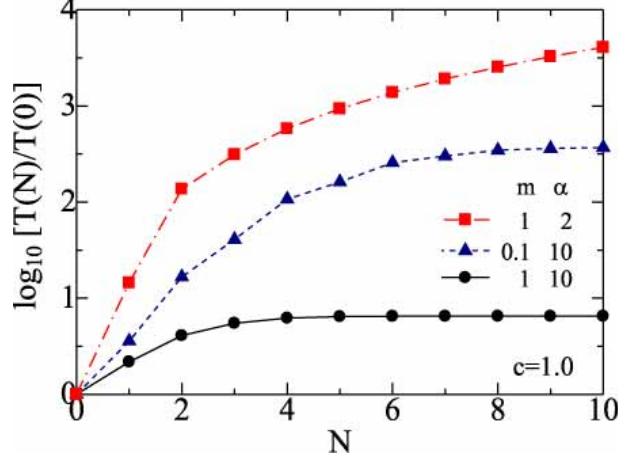


FIG. 16: (Color online) The N dependence of the tunneling period T for three sets of parameters of $(m, \alpha) = (1.0, 10.0)$ (solid curve), $(0.1, 10.0)$ (dashed curve) and $(1.0, 2.0)$ (chain curve) for a linear coupling ($d = 1$) with $c = 1.0$. Results of $N = 0$ stand for those of no couplings.

$N_m + 1 = 2(N + 1)$. Since we have neglected excited states higher than ϵ_2 of DW system, a reasonable choice of N for the maximum quantum state of HO is expected to be given by

$$\left(N + \frac{1}{2}\right) \hbar\omega \sim (\epsilon_2 - \epsilon_0) = 3.4641. \quad (81)$$

By using Eqs. (27) and (81), we obtain $N \sim 3.5$ for $\alpha = 10.0$ and $N \sim 19.6$ for $\alpha = 2.0$.

N dependences of the tunneling period T calculated for three sets of parameters: $(m, \alpha) = (1.0, 10.0)$, $(0.1, 10.0)$ and $(1.0, 2.0)$ with $c = 1.0$ are shown in Fig. 16 where the ordinate is in the logarithmic scale. It is noted that T is significantly increased with increasing N , in particular for smaller m and smaller α . In the case of $(m, \alpha) = (1.0, 10.0)$, the enhancement of T saturates at $N \gtrsim 4$. On the contrary, such a saturation is not realized in the case of $(m, \alpha) = (1.0, 2.0)$ even at $N = 10$.

Paying attention to the case of $(m, \alpha) = (1.0, 2.0)$ which shows the most significant N dependence of T in Fig. 16, we have calculated the time-dependent wavepackets for $N = 1$ and 5, whose results are shown in Fig. 17. We obtain $\Omega_1 = 0.594662 \times 10^{-3}$ for $N = 1$, and $\Omega_1 = 0.923365 \times 10^{-4}$ for $N = 5$. The initial position of wavepacket for $N = 1$ is $(x, y) = (1.23526, 1.66213)$, while that for $N = 5$ is $(x, y) = (1.23532, 5.54669)$. Figure 17(a)-17(c) show magnitudes of wavepackets at $0.0 \leq t \leq T_1/2$ where $T_1 (= 1056.6)$ denotes the tunneling period for $N = 1$. Figure 17(d)-17(f) show similar results at $0.0 \leq t \leq T_5/2$ where the tunneling period for $N = 5$ is $T_5 = 68046.6$. Comparing time-dependent magnitudes of

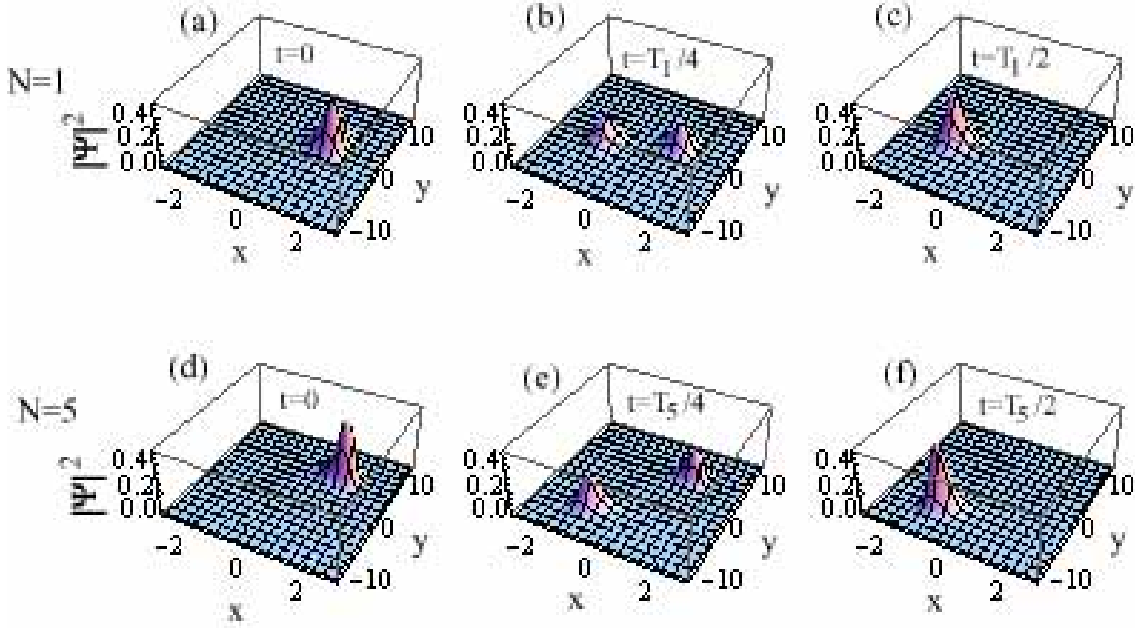


FIG. 17: (Color online) $|\Psi(x, y, t)|^2$ with $N = 1$ at (a) $t = 0.0$, (b) $t = T_1/4$ and (c) $t = T_1/2$ where $T_1 = 1056.6$; $|\Psi(x, y, t)|^2$ with $N = 5$ at (d) $t = 0.0$, (e) $t = T_5/4$ and (f) $t = T_5/2$ where $T_5 = 68046.6$ ($d = 1$, $m = 1.0$, $\alpha = 2.0$ and $c = 1.0$).

wavepackets in Figs. 17(a)-17(c) for $N = 1$ with those in Fig. 17(d)-17(f) for $N = 5$, we note that two results are similar when reading them by the normalized time t/T , despite the fact that the tunneling period T_5 is larger than T_1 by a factor of 64.4.

D. Uncertainty relation

The Heisenberg uncertainty of $\Delta x \Delta p_x$, which is also a typical quantum phenomenon, is related with the tunneling [10]. We may obtain analytical expressions for averages of fluctuations of x and p_x in the case of $N = 1$. For a linear coupling, we obtain

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2, \quad (82)$$

$$= \frac{\gamma_0}{2}(\cos^2 \theta_1 + \sin^2 \theta_2) + \frac{\gamma_1}{2}(\sin^2 \theta_1 + \cos^2 \theta_2) - \gamma^2 \cos^2(\theta_1 - \theta_2) \cos^2 \Omega_1 t, \quad (83)$$

$$(\Delta p_x)^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2, \quad (84)$$

$$= \frac{\chi_0}{2}(\cos^2 \theta_1 + \sin^2 \theta_2) + \frac{\chi_1}{2}(\sin^2 \theta_1 + \cos^2 \theta_2) - \eta^2 \cos^2(\theta_1 + \theta_2) \sin^2 \Omega_1 t, \quad (85)$$

with

$$\chi_0 = \int_{-\infty}^{\infty} [\partial_x \phi_0(x)]^2 dx = 2.58707, \quad (86)$$

$$\chi_1 = \int_{-\infty}^{\infty} [\partial_x \phi_1(x)]^2 dx = 3.17399, \quad (87)$$

where θ_1 and θ_2 are given by Eqs. (41) and (42), respectively. For a quadratic coupling, they are given by

$$(\Delta x)^2 = \frac{\gamma_0}{2} + \frac{\gamma_1}{2} - \gamma^2 \cos^2(\theta_1 - \theta_2) \cos^2 \Omega_1 t, \quad (88)$$

$$(\Delta p_x)^2 = \frac{\chi_0}{2} + \frac{\chi_1}{2} - \eta^2 \cos^2(\theta_1 - \theta_2) \sin^2 \Omega_1 t, \quad (89)$$

where θ_1 and θ_2 are given by Eqs. (68) and (69), respectively. For uncoupled DW ($c = 0.0$) where $\theta_1 = \theta_2 = 0$, Eqs. (83) and (85) reduce to

$$(\Delta x)^2 = \frac{\gamma_0}{2} + \frac{\gamma_1}{2} - \gamma^2 \cos^2 \Omega_1 t, \quad (90)$$

$$(\Delta p_x)^2 = \frac{\chi_0}{2} + \frac{\chi_1}{2} - \eta^2 \sin^2 \Omega_1 t. \quad (91)$$

Figure 18(a) shows time dependences of Δx and Δp_x for a linear coupling with $m = 1.0$, $\alpha = 10.0$, $c = 1.0$ and $N = 1$. Although Δx oscillates with an appreciable magnitude of γ^2 in Eq. (83), Δp_x is almost constant ($\simeq 1.69$) because of a small η^2 in Eq. (85). Figure 18(b) shows the uncertainty given by $\Delta x \Delta p_x$, which is initially 0.556875. The Heisenberg uncertainty relation: $\Delta x \Delta p_x \geq \hbar/2$ is always preserved. We note that $\Delta x \Delta p_x$ has a large magnitude at $t \sim T/4$ or $3T/4$ when tunneling takes place (Fig. 5). This shows that the uncertainty is related with quantum tunneling [10].

V. CONCLUDING REMARK

We have studied wavepacket dynamics in the Razavy hyperbolic DW potential [6] which is coupled to a HO by linear and quadratic interactions. Wavepackets show the quantum tunneling between two bottoms in the composite potential $U(x, y)$ [Eq. (31)]. The tunneling period is increased with increasing c and/or N , which is more significant for smaller m and smaller α (Figs. 15 and 16). Comparing results of linear and quadratic couplings, we note that the tunneling probability $P_r(t)$ is the same and the marginal probability density $\rho_x(t)$ is similar between the two, but $\rho_y(t)$ is different (Figs. 6 and 10). Furthermore, $\langle y \rangle$ vs. $\langle p_y \rangle$

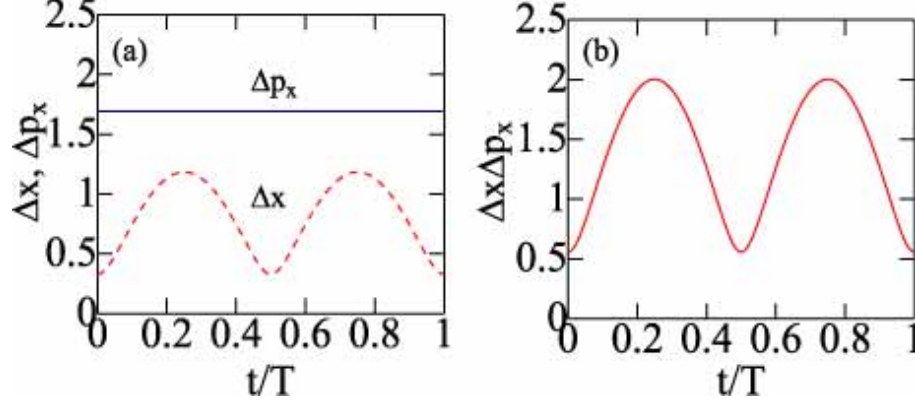


FIG. 18: (Color online) (a) Time dependences of Δx (dashed curve) and Δp_x (solid curve), and (b) the uncertainty: $\Delta x \Delta p_x$ for a linear coupling ($m = 1.0$, $\alpha = 1.0$, $c = 1.0$ and $N = 1$).

plot in the quadratic coupling stay at the initial position in the phase space, while that in the linear coupling shows an elliptic motion (Figs. 7 and 11). Our phase space plots of $\langle x \rangle$ vs. $\langle p_x \rangle$ and $\langle y \rangle$ vs. $\langle p_y \rangle$ for the two-term wavepacket given by Eq. (25) are quite different from relevant results obtained in [5], where trajectories show ellipse-like orbits but they do not return to initial positions after revolution in a quartic DW potential coupled to HO (see Figs. 4.2 and 4.5 in [5]). Ref. [5] showed that this oddity occurs even for uncoupled DW (see Fig. 3.2 in [5]), for which our calculation leads to the complete elliptic trajectory for $\langle x \rangle$ vs. $\langle p_x \rangle$ plot because we obtain

$$\langle x \rangle = \gamma \cos \Omega_1 t, \quad \langle p_x \rangle = -\eta \sin \Omega_1 t, \quad (92)$$

for $c = 0.0$ in Eqs. (51) and (53). This difference between the result of Ref.[5] and ours does not arise from the difference between quartic and hyperbolic DW potentials, because a chain of equations of motion for expectation values in a general symmetric DW potential is closed within $\langle x \rangle$ and $\langle p_x \rangle$ for the two-term wavepacket (see the Appendix). It has been shown that the uncertainty of $\Delta x \Delta p_x$ becomes appreciable when the tunneling takes place (Fig. 18). It would be interesting to experimentally observed $|\Psi(x, t)|^2$ and $\Delta x \Delta p_x$, which might be possible with advanced recent technology.

Acknowledgments

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Appendix: Expectation values for the two-term wavepacket

A coupled DW system is assumed to be described by the Hamiltonian given by

$$H = \frac{p_x^2}{2M} + \frac{p_y^2}{2m} + U(x, y), \quad (\text{A1})$$

with

$$U(x, y) = V(x) + \frac{m\omega^2 y^2}{2} - c x^d y, \quad (\text{A2})$$

where $V(x)$ denotes a general symmetric DW potential and $d = 1$ ($d = 2$) signifies a linear (quadratic) coupling. We consider the two-term wavepacket given by

$$\Psi(x, y, t) = \frac{1}{\sqrt{2}} [\Phi_0(x, y) e^{-iE_0 t/\hbar} + \Phi_1(x, y) e^{-iE_1 t/\hbar}], \quad (\text{A3})$$

where real eigenfunctions of $\Phi_0(x, y)$ and $\Phi_1(x, y)$ satisfy the Schödinger equation

$$H\Phi_0(x, y) = E_0\Phi_0(x, y), \quad (\text{A4})$$

$$H\Phi_1(x, y) = E_1\Phi_1(x, y) = (E_0 + \Omega_1)\Phi_1(x, y). \quad (\text{A5})$$

Expectation values of $\langle x \rangle$ and $\langle p_x \rangle$ for the wavepacket are expressed by

$$\langle x \rangle = a_x \cos \Omega_1 t, \quad (\text{A6})$$

$$\langle p_x \rangle = -b_x \sin \Omega_1 t, \quad (\text{A7})$$

where

$$a_x = \int \int \Phi_0(x, y) x \Phi_1(x, y) dx dy, \quad (\text{A8})$$

$$b_x = \int \int \Phi_0(x, y) (-i\partial_x) \Phi_1(x, y) dx dy. \quad (\text{A9})$$

Equations (A6) and (A7) lead to

$$\frac{d\langle x \rangle}{dt} = -\Omega_1 a_x \sin \Omega_1 t = \left(\frac{\Omega_1 a_x}{b_x} \right) \langle p_x \rangle, \quad (\text{A10})$$

$$\frac{d\langle p_x \rangle}{dt} = -\Omega_1 b_x \cos \Omega_1 t = -\left(\frac{\Omega_1 b_x}{a_x} \right) \langle x \rangle. \quad (\text{A11})$$

On the other hand, Heisenberg equations of motion for x and p_x are given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{p_x}{M}, \quad (\text{A12})$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial x} = -\partial_x U(x, y). \quad (\text{A13})$$

Taking averages of Eqs. (A12) and (A13) over the two-term wavepacket $\Psi(x, y, t)$ given by Eq. (A3), we obtain

$$\frac{d\langle x \rangle}{dt} = -\frac{1}{M} \int \int \Phi_0(x, y) \partial_x \Phi_1(x, y) dx dy \sin \Omega_1 t, \quad (\text{A14})$$

$$\frac{d\langle p_x \rangle}{dt} = - \int \int \Phi_0(x, y) \partial_x U(x, y) \Phi_1(x, y) dx dy \cos \Omega_1 t. \quad (\text{A15})$$

The equivalence of Eqs. (A10) and (A11) with Eqs. (A14) and (A15), respectively, may be shown as follows: Multiplying Eqs. (A4) and (A5) by $x \Phi_1(x, y)$ and integrating them over x and y with integrations by parts, we obtain

$$\Omega_1 a_x = \frac{1}{M} \int \int \Phi_0(x, y) \partial_x \Phi_1(x, y) dx dy. \quad (\text{A16})$$

Multiplications of Eqs. (A4) and (A5) by $\partial_x \Phi_1(x, y)$ and integrations of them over x and y lead to

$$\Omega_1 b_x = \int \int \Phi_0(x) \partial_x U(x, y) \Phi_1(x, y) dx dy. \quad (\text{A17})$$

It is well known that Heisenberg equations of motion given by Eqs. (A12) and (A13) generally yield a hierarchical chain for DW potentials. Fortunately, equations of motion for expectation values for the two-term wavepacket close within $\langle x \rangle$ and $\langle p_x \rangle$ as given by Eqs. (A10) and (A11). Such a simplification does not occur for a general wavepacket, as given by Eqs. (23) and (24).

Similar calculations may be made also for $\langle y \rangle$ and $\langle p_y \rangle$. Then both the $\langle x \rangle$ vs. $\langle p_x \rangle$ plot and the $\langle y \rangle$ vs. $\langle p_y \rangle$ plot are elliptic for the two-term wavepacket.

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